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GENERALIZATIONS OF JENSEN-MERCER'S INEQUALITY

ZLATKO PAVIĆ

Mechanical Engineering Faculty in Slavonski Brod
University of Osijek
Trg Ivane Brlić Mažuranić 2
35000 Slavonski Brod
Croatia
e-mail: Zlatko.Pavic@sfsb.hr

Abstract

The article deals with the generalizations of Jensen-Mercer's inequality using affine combinations, which can be represented as convex combinations. The generalized Jensen-Mercer's inequality is also obtained for the convex function of several variables applying affine combinations of the simplex.

1. Introduction

1.1. Combinations of scalars and vectors

The convex sets are generally observed in a real vector space $\mathcal{X}$. Affiliation to some vector set is analytically expressed by combinations of vectors (points) $x_i \in \mathcal{X}$ and scalars (coefficients) $p_i \in \mathbb{R}$. The sum

$$\sum_{i=1}^{n} p_i x_i, \tag{1.1}$$

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belongs to the vector subspace $\mathrm{lin}\{x_i\}$ (the smallest vector space that contains all x_i), and it is called the linear combination. If $\sum_{i=1}^{n} p_i = 1$, the sum in (1.1) belongs to the affine hull $\mathrm{aff}\{x_i\}$ (the smallest translated vector space that contains all x_i), and it is called the affine combination. If $\sum_{i=1}^{n} p_i = 1$ and all $p_i \in [0, 1]$, the sum in (1.1) belongs to the convex hull $\mathrm{conv}\{x_i\}$ (the smallest convex vector set that contains all x_i), and it is called the convex combination.

1.2. Jensen's inequality and related results

In the discrete case, Jensen's inequality is applied to the convex combinations of vectors.

Theorem A [Jensen's inequality]**.** *Let $\mathcal{X}$ be a real vector space. Let $\sum_{i=1}^{n} p_i x_i$ be a convex combination of vectors $x_i \in \mathcal{X}$ and scalars $p_i \in \mathbb{R}$ of the sum $\sum_{i=1}^{n} p_i = 1$. Then every convex function $f : \mathrm{conv}\{x_1, \dots, x_n\} \to \mathbb{R}$ verifies the inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{1.2}$$

The famous inequality in (1.2) is usually called the discrete form of Jensen's inequality, and can be found in many books, for example, in [5, pages 112-113].

Let us mention the following two related results:

Theorem B [3, Theorem 1.2]**.** *Let $x_1, \dots, x_n \in [a, b] \subset \mathbb{R}$ be points, and $p_i \geq 0$ be weights with $\sum_{i=1}^{n} p_i = 1$. If $f : [a, b] \to \mathbb{R}$ is a convex function, then*

$$f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) \leq f(a) + f(b) - \sum_{i=1}^{n} p_i f(x_i). \tag{1.3}$$

The inequality in (1.3) is usually called Jensen-Mercer's inequality. Generalization of this inequality turned toward Jensen-Steffensen's inequality has been achieved in [1]. Jensen-Steffensen's inequality uses the assumption of order of the points x_i.

The Jensen-Mercer inequality in (1.3) was generalized by applying the majorization assumptions.

Theorem C [6, Theorem 2.1]. *Let* $f : \mathcal{I} \to \mathbb{R}$ *be a continuous convex function on an interval* $\mathcal{I} \subset \mathbb{R}$. *Suppose* $\boldsymbol{a} = (a_1, \ldots, a_m)$ *with* $a_j \in \mathcal{I}$, *and* $\boldsymbol{X} = (x_{ij})$ *is a real* $n \times m$ *matrix such that* $x_{ij} \in \mathcal{I}$ *for all* i, j. *Let* $p_i \geq 0$ *be weights with* $\sum_{i=1}^{n} p_i = 1$.

If $\boldsymbol{a}$ *majorizes each row of* $\boldsymbol{X}$, *that is,*

$$\boldsymbol{x}_i = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \boldsymbol{a} \text{ for each } i = 1, \ldots, n,$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} p_i x_{ij}\right) \leq \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} p_i f(x_{ij}). \tag{1.4}$$

2. Inequalities for Convex Functions of One Variable

The main result in this section is Theorem 2.4, which represents Jensen's inequality for one variable convex function and affine combinations from the interval.

In what follows, we use a real interval $[a, b]$ assuming $a < b$. Every $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \alpha_x a + \beta_x b, \tag{2.1}$$

where

$$\alpha_x = \frac{\begin{vmatrix} x & 1 \\ b & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}}, \quad \beta_x = -\frac{\begin{vmatrix} x & 1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}}. \tag{2.2}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and $f_{\{a,b\}}^{\text{cho}}$ be the chord line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. If $x \in [a, b]$, then the above combination is convex, and we have the chord inequality

$$f(x) \le \alpha_x f(a) + \beta_x f(b) = f_{\{a,b\}}^{\text{cho}}(x). \tag{2.3}$$

If $x \notin [a, b]$, then the reverse inequality is valid in (2.3).

Simplicity of the chord line $f_{\{a,b\}}^{\text{cho}}(x)$ as the affine function $h(x) = kx + l$ will be very useful.

Lemma 2.1. *Let* $x_i \in \mathbb{R}$ *be points. Let* $\alpha_i \in \mathbb{R}$ *be coefficients of the sum* $\sum_{i=1}^{n} \alpha_i = 1$. *Then every affine function* $h : \mathbb{R} \to \mathbb{R}$ *verifies the equality*

$$h\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i h(x_i). \tag{2.4}$$

We will examine the behaviour of convex functions on some special types of affine combinations.

Lemma 2.2. *Let* $c \in [a, b]$ *be a point. Let* $\alpha, \beta \in [0, 1]$, $\gamma \in [-1, 1]$ *be coefficients of the sum* $\alpha + \beta + \gamma = 1$. *Then every convex function* $f : [a, b] \to \mathbb{R}$ *verifies the inequality*

$$f(\alpha a + \beta b + \gamma c) \leq \alpha f(a) + \beta f(b) + \gamma f(c). \tag{2.5}$$

Proof. First of all, let us show that the affine combination $\alpha a + \beta b + \gamma c$ belongs to $[a, b]$. Since $c \in [a, b]$, it has to be $c = \alpha_c a + \beta_c b$ for coefficients α_c and β_c taken from the formulas in (2.2). Then we have

$$\begin{aligned} \alpha a + \beta b + \gamma c &= \alpha a + \beta b + (1 - \alpha - \beta)(\alpha_c a + \beta_c b) \\ &= [\alpha(1 - \alpha_c) + (1 - \beta)\alpha_c]a + [\beta(1 - \beta_c) + (1 - \alpha)\beta_c]b. \end{aligned} \tag{2.6}$$

The coefficients in the square brackets are non-negative with the sum equal to 1, so the observed expression $\alpha a + \beta b + \gamma c$ belongs to $[a, b]$.

If $\gamma \geq 0$, the inequality in (2.5) is the Jensen inequality for the three-membered convex combination $\alpha a + \beta b + \gamma c$.

If $\gamma \leq 0$, we use the inequality in (2.3) and the affinity of the chord line $f^{\text{cho}}_{\{a, b\}}$, in this way;

$$\begin{aligned} f(\alpha a + \beta b + \gamma c) &\leq f^{\text{cho}}_{\{a, b\}}(\alpha a + \beta b + \gamma c) \\ &= \alpha f^{\text{cho}}_{\{a, b\}}(a) + \beta f^{\text{cho}}_{\{a, b\}}(b) + \gamma f^{\text{cho}}_{\{a, b\}}(c) \\ &\leq \alpha f(a) + \beta f(b) + \gamma f(c), \end{aligned}$$

respecting that $f^{\text{cho}}_{\{a, b\}}(a) = f(a)$, $f^{\text{cho}}_{\{a, b\}}(b) = f(b)$, and $\gamma f^{\text{cho}}_{\{a, b\}}(c) \leq \gamma f(c)$. $\square$

Sufficient conditions on the coefficients in Lemma 2.2 are: $\alpha, \beta \in [0, 1]$ and $\alpha + \beta + \gamma = 1$. From these conditions, it follows $\gamma \in [-1, 1]$.

The following useful fact is the consequence of the equality in (2.6).

Corollary 2.3. *An affine combination* $\alpha a + \beta b + \gamma c$ *belongs to* $[a, b]$ *for every* $c \in [a, b]$ *if and only if the coefficients* α *and* β *belong to* $[0, 1]$.

Proof. The sufficiency is proved in Lemma 2.2. The necessity follows by putting the extreme values of the coefficients α_c and β_c (0 or 1) in the right-hand side of the equality in (2.6). □

The analytic inequality written in the formula in (2.5) can be described by a geometric figure. Given $c \in [a, b]$, take the graph points $A(a, f(a))$, $B(b, f(b))$, and $C(c, f(c))$, and determine the position of the points

$$P(\alpha a + \beta b + \gamma c,\ \alpha f(a) + \beta f(b) + \gamma f(c)),$$

for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta + \gamma = 1$. We have the radius-vectors equality

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C = (\alpha + \gamma)\vec{r}_A + (\beta + \gamma)\vec{r}_B - \gamma(\vec{r}_A + \vec{r}_B - \vec{r}_C). \tag{2.7}$$

If $\gamma \geq 0$, the left-hand side of the equality in (2.7) represents the convex combinations of the vectors $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$. Interpreted geometrically, the points P belong to the triangle $\operatorname{conv}\{A, B, C\}$.

If $\gamma \leq 0$, the coefficients $\alpha + \gamma, \beta + \gamma, -\gamma \in [0, 1]$ with the sum equal to 1. Take the point $D(a + b - c,\ f(a) + f(b) - f(c))$. The right-hand side of the equality in (2.7) represents the convex combinations of the vectors $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_D$. This means that the geometric location of the points P is just the triangle $\operatorname{conv}\{A, B, D\}$.

Theorem 2.4. *Let* $x_i \in [a, b]$ *be points. Let* $\alpha, \beta, p_i \in [0, 1]$, $\gamma \in [-1, 1]$ *be coefficients of the sums* $\alpha + \beta + \gamma = \sum_{i=1}^{n} p_i = 1$. *Then every convex function* $f : [a, b] \to \mathbb{R}$ *verifies the inequality*

$$f\left(\alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i\right) \leq \alpha f(a) + \beta f(b) + \gamma \sum_{i=1}^{n} p_i f(x_i). \tag{2.8}$$

Proof. Since $\sum_{i=1}^{n} p_i x_i \in [a, b]$, the affine combination $\alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i \in [a, b]$ by Lemma 2.2. The proof continues as in Lemma 2.2. Particularly, in the case $\gamma \leq 0$, the chord inequality $\gamma f_{\{a, b\}}^{\text{cho}}(x_i) \leq \gamma f(x_i)$ is applied to every x_i. $\square$

For $\alpha = \beta = 0$ and $\gamma = 1$, the inequality in (2.8) is reduced to the Jensen inequality. For $\alpha = \beta = 1$ and $\gamma = -1$, the inequality in (2.8) is reduced to the Jensen-Mercer inequality.

Given a function f, points x_i and coefficients $\alpha, \beta, \gamma, p_i$ that satisfy the conditions of Theorem 2.4, the points

$$P\left(\alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i,\ \alpha f(a) + \beta f(b) + \gamma \sum_{i=1}^{n} p_i f(x_i)\right),$$

belong to the convex polygon

$$\mathcal{P} = \operatorname{conv}\{A(a, f(a)), C_i(x_i, f(x_i)), D_i(a + b - x_i, f(a) + f(b) - f(x_i)), B(b, f(b))\}.$$

Such a polygon $\mathcal{P}$ for $n = 2$ is shown in Figure 1.

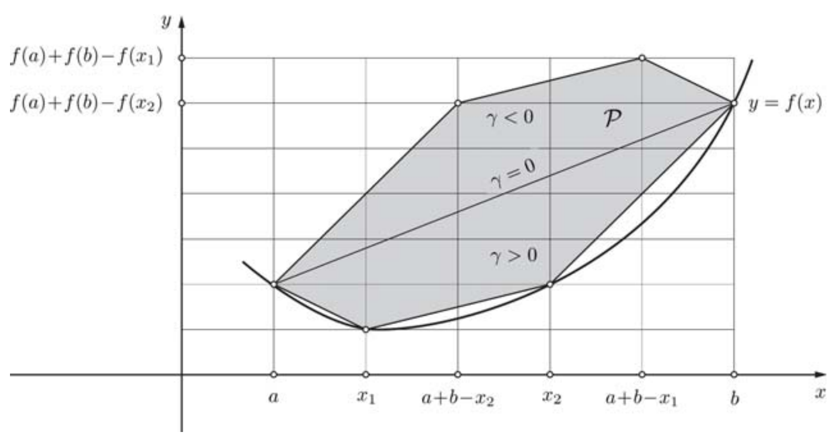

Figure 1. Geometric interpretation of the inequality in (2.8).

Lemma 2.2 can be used to obtain a more general variant of Theorem 2.4 introducing points a_j that do not belong to the convex hull of the points x_i, as stated:

Theorem 2.5. *Let* $a_j,\ x_i \in \mathbb{R}$ *be points so that*

$$\{a_1, \dots, a_m\} \cap \operatorname{conv}\{x_1, \dots, x_n\} = \emptyset \ \textit{or}\ \{\textit{endpoint}(s)\}.$$

Let $\alpha_j,\ p_i \in [0, 1],\ \gamma \in [-1, 1]$ *be coefficients of the sums* $\sum_{j=1}^{m} \alpha_j + \gamma = \sum_{i=1}^{n} p_i = 1.$

If

$$\frac{1}{\sum_{j=1}^{m}\alpha_j} \sum_{j=1}^{m} \alpha_j a_j \in \operatorname{conv}\{x_1, \dots, x_n\} \ \textit{for}\ \sum_{j=1}^{m} \alpha_j > 0,$$

then every convex function $f : \operatorname{conv}\{a_1, \dots, a_m\} \to \mathbb{R}$ *verifies the inequality*

$$f\left(\sum_{j=1}^{m} \alpha_j a_j + \gamma \sum_{i=1}^{n} p_i x_i\right) \le \sum_{j=1}^{m} \alpha_j f(a_j) + \gamma \sum_{i=1}^{n} p_i f(x_i). \tag{2.9}$$

Proof. Take $[a, b] = \operatorname{conv}\{x_1, \dots, x_n\}$. The representation $\sum_{j=1}^{m} \alpha_j a_j = \alpha a + \beta b$ can be achieved with $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = \sum_{j=1}^{m} \alpha_j$. If we still substitute $c = \sum_{i=1}^{n} p_i x_i$, the combination under the function f is reduced to the affine combination $\alpha a + \beta b + \gamma c \in [a, b]$.

If $a < b$, the proof of Lemma 2.2 can be applied, respecting the inequalities $\alpha_j f_{\{a,b\}}^{\text{cho}}(a_j) \le \alpha_j f(a_j)$, and $\gamma f_{\{a,b\}}^{\text{cho}}(x_i) \le \gamma f(x_i)$ for $\gamma \le 0$.

If $a = b$, the support line $y = f_{\{a\}}^{\text{sup}}(x)$ would be used. □

Similar assumptions as in Theorem 2.5 with the internal and external points were applied to obtain Jensen's inequality in [7, Section 2].

We are trying to generalize the inequality in (2.8) using the special affine combinations $\sum_{i=1}^{n} \gamma_i x_i$ instead of the convex combinations $\sum_{i=1}^{n} p_i x_i$. As the orientation, the following lemma can be specified.

Lemma 2.6. *Let* $c, d \in [a, b]$ *be points. Let* $\alpha, \beta, \gamma \in [0, 1]$, $\delta \in [-1, 1]$ *be coefficients of the sum* $\alpha + \beta + \gamma + \delta = 1$.

If $\alpha + \delta \geq 0$ *and* $\beta + \delta \geq 0$, *then every convex function* $f : [a, b] \to \mathbb{R}$ *verifies the inequality*

$$f(\alpha a + \beta b + \gamma c + \delta d) \leq \alpha f(a) + \beta f(b) + \gamma f(c) + \delta f(d). \tag{2.10}$$

Proof. Let us prove the inequality in (2.10) for $\delta \leq 0$. Since the right-hand side of the equality

$$\alpha a + \beta b + \gamma c + \delta d = (\alpha + \delta)a + (\beta + \delta)b + \gamma c - \delta(a + b - d), \tag{2.11}$$

is the convex combination, we first apply Jensen's inequality, and then the inequality in (2.5) to $a + b - d$. The procedure goes as follows:

$$\begin{aligned} f(\alpha a + \beta b + \gamma c + \delta d) &\leq (\alpha + \delta)f(a) + (\beta + \delta)f(b) + \gamma f(c) - \delta f(a + b - d) \\ &\leq (\alpha + \delta)f(a) + (\beta + \delta)f(b) + \gamma f(c) - \delta[f(a) + f(b) - f(d)] \\ &= \alpha f(a) + \beta f(b) + \gamma f(c) + \delta f(d), \end{aligned}$$

which ends the proof of the observed case. □

It is not necessary to require the condition $\delta \in [-1, 1]$, because it follows from other coefficient conditions.

Applying experience of Lemma 2.6, we can formulate the following generalization of Theorem 2.4:

Theorem 2.7. *Let* $x_i \in [a, b]$ *be points. Let* $\alpha, \beta \in [0, 1]$, $\gamma, \gamma_i \in [-1, 1]$ *be coefficients of the sums* $\alpha + \beta + \gamma = \sum_{i=1}^{n} \gamma_i = 1$. *Suppose* $\gamma_{i_1}, \ldots, \gamma_{i_k} \geq 0$ *and* $\gamma_{i_{k+1}}, \ldots, \gamma_{i_n} \leq 0$. *Let* $\delta = \sum_{j=1}^{k} \gamma_{i_j}$ *and* $\bar{\delta} = \sum_{j=k+1}^{n} \gamma_{i_j}$.

If either

$$\gamma \geq 0, \quad \alpha + \gamma\bar{\delta} \geq 0, \quad \beta + \gamma\bar{\delta} \geq 0,$$

or

$$\gamma \leq 0, \quad \alpha + \gamma\delta \geq 0, \quad \beta + \gamma\delta \geq 0,$$

then every convex function $f : [a, b] \to \mathbb{R}$ *verifies the inequality*

$$f\left(\alpha a + \beta b + \gamma \sum_{i=1}^{n} \gamma_i x_i\right) \leq \alpha f(a) + \beta f(b) + \gamma \sum_{i=1}^{n} \gamma_i f(x_i). \tag{2.12}$$

Proof. Let us prove the case $\gamma \geq 0$. The right-hand side of the decomposition

$$\begin{aligned} \alpha a + \beta b + \gamma \sum_{i=1}^{n} \gamma_i x_i &= \alpha a + \beta b + \gamma \sum_{j=1}^{k} \gamma_{i_j} x_{i_j} + \gamma \sum_{j=k+1}^{n} \gamma_{i_j} x_{i_j} \\ &= (\alpha + \gamma\bar{\delta})a + (\beta + \gamma\bar{\delta})b + \gamma \sum_{j=1}^{k} \gamma_{i_j} x_{i_j} \\ &\quad + \gamma \sum_{j=k+1}^{n} (-\gamma_{i_j})(a + b - x_{i_j}), \end{aligned}$$

is the convex combination that belongs to $[a, b]$. We first apply the Jensen inequality to the above convex combination, then the inequality in (2.5) to affine combinations $a + b - x_{ij}$, and so, using the same procedure as in Lemma 2.6, obtain the inequality in (2.12). □

The inequality in (2.12), together with its assumptions, recalls the Jensen-Steffensen inequality, but no requirement for the order of the points x_i.

2.1. Generalizations

Theorem 2.4 can be generalized by introducing a real valued function g defined on the non-empty set $\mathcal{S}$, as follows:

Corollary 2.8. *Let* $g : \mathcal{S} \to \mathbb{R}$ *be a function that attains extreme values* $g(a)$, $g(b)$ *for some* $a, b \in \mathcal{S}$, *thus* $g(x) \in \operatorname{conv}\{g(a), g(b)\} = \mathcal{I}$ *for every* $x \in \mathcal{S}$. *Let* $x_i \in \mathcal{S}$ *be points. Let* $\alpha, \beta, p_i \in [0, 1], \gamma \in [-1, 1]$ *be coefficients of the sums* $\alpha + \beta + \gamma = \sum_{i=1}^{n} p_i = 1$. *Then the inequality*

$$f\left(\alpha g(a) + \beta g(b) + \gamma \sum_{i=1}^{n} p_i g(x_i)\right) \leq \alpha f(g(a)) + \beta f(g(b)) + \gamma \sum_{i=1}^{n} p_i f(g(x_i)), \tag{2.13}$$

holds for every convex function $f : \mathcal{I} \to \mathbb{R}$.

We can include more functions and more intervals in the generalization procedure.

Corollary 2.9. *Let* $g_j : \mathcal{S} \to \mathbb{R}$ *be functions that attain extreme values* $g_j(a_j)$, $g_j(b_j)$ *for some* $a_j, b_j \in \mathcal{S}$, *thus* $g_j(x) \in \operatorname{conv}\{g_j(a), g_j(b)\} = \mathcal{I}_j$ *for every* $x \in \mathcal{S}$. *Let* $x_{i_j} \in \mathcal{S}$ *be points. Let* $\alpha_j, \beta_j, p_{ij}, p_j \in [0, 1]$, $\gamma_j \in [-1, 1]$ *be coefficients of the sums* $\alpha_j + \beta_j + \gamma_j = \sum_{i=1}^{n} p_{ij} = \sum_{j=1}^{m} p_j = 1$. *Then the inequality*

$$f\left(\sum_{j=1}^{m} p_j\left(\alpha_j g_j(a_j) + \beta_j g_j(b_j) + \gamma_j \sum_{i=1}^{n} p_{ij} g_j(x_{ij})\right)\right)$$

$$\leq \sum_{j=1}^{m} p_j\left(\alpha_j f(g_j(a_j)) + \beta_j f(g_j(b_j)) + \gamma_j \sum_{i=1}^{n} p_{ij} f(g_j(x_{ij}))\right), \tag{2.14}$$

holds for every convex function $f : \operatorname{conv}\{\mathcal{I}_1 \cup \ldots \cup \mathcal{I}_m\} \to \mathbb{R}$.

Proof. First apply the basic form of Jensen's inequality with respect to the non-negative coefficients p_j, then Corollary 2.8. □

The generalizations of Theorem 2.7 can be implemented in exactly the same way.

2.2. Application to quasi-arithmetic means

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. In the applications of convexity, we often use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (ψ is φ-convex), that is, $f = \psi \circ \varphi^{-1}$ is convex on $\varphi(\mathcal{I})$. A similar notation is used for the concavity.

Let $\sum_{i=1}^{n} p_i x_i$ be a convex combination from $\mathcal{I}$. The discrete φ-quasi-arithmetic mean of the points x_i with the coefficients p_i is the point

$$M_{\varphi}(x_i;\ p_i) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i)\right), \tag{2.15}$$

which belongs to $\mathcal{I}$. The point $M_{\varphi}(x_i;\ p_i)$ can also be called the φ-quasi-center of the convex combination $c = \sum_{i=1}^{n} p_i x_i$. The formula in (2.15) may be applied for a quasi-arithmetic mean definition of an affine combination $\alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i$ that belongs to $[a, b]$, in this way;

$$M_{\varphi}(a,\ b,\ x_i;\ \alpha,\ \beta,\ \gamma p_i) = \varphi^{-1}\left(\alpha\varphi(a) + \beta\varphi(b) + \gamma\sum_{i=1}^{n} p_i \varphi(x_i)\right). \tag{2.16}$$

The mean defined in (2.16) belongs to $[a, b]$ because $\alpha\varphi(a) + \beta\varphi(b) + \gamma \sum_{i=1}^{n} p_i \varphi(x_i)$ belongs to $\varphi([a, b])$.

We have the following application of Theorem 2.4 to the quasi-arithmetic means.

Corollary 2.10. *Let* $x_i \in [a, b]$ *be points. Let* $p_i, \alpha, \beta \in [0, 1]$, $\gamma \in [-1, 1]$ *be coefficients of the sums* $\alpha + \beta + \gamma = \sum_{i=1}^{n} p_i = 1$. *Let* $\varphi, \psi : [a, b] \to \mathbb{R}$ *be strictly monotone continuous functions.*

If ψ *is either* φ*-convex and increasing or* φ*-concave and decreasing, then the inequality*

$$M_{\varphi}(a, b, x_i; \alpha, \beta, \gamma p_i) \leq M_{\psi}(a, b, x_i; \alpha, \beta, \gamma p_i), \tag{2.17}$$

holds.

If ψ *is either* φ*-convex and decreasing or* φ*-concave and increasing, then the reverse inequality is valid in* (2.17).

Proof. The proof is easy and uses the convex (respectively, concave) function $f = \psi \circ \varphi^{-1} : \varphi([a, b]) \to \mathbb{R}$, which can be applied to Theorem 2.4. □

Using the pairs of functions $\varphi(x) = x^{-1}$, $\psi(x) = \ln x$, and $\varphi(x) = \ln x$, $\psi(x) = x$ in the inequality in (2.17) with $a, b > 0$, we get the harmonic-geometric-arithmetic inequality for the means defined in (2.16):

$$\left(\frac{\alpha}{a} + \frac{\beta}{b} + \gamma \sum_{i=1}^{n} \frac{p_i}{x_i}\right)^{-1} \leq a^{\alpha} b^{\beta} \prod_{i=1}^{n} x_i^{\gamma p_i} \leq \alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i. \tag{2.18}$$

The inequality in (2.17) can be refined using the procedures given in [4, Section 2]. Theorem 2.7, Corollaries 2.8 and 2.9 can also be applied to derive quasi-arithmetic means.

3. Inequalities for Convex Functions of Several Variables

The main result in this section is Theorem 3.2, which represents Jensen's inequality for two variables convex function and affine combinations from the triangle.

Take the three planar points $A(x_A,\ y_A)$, $B(x_B,\ y_B)$, and $C(x_C,\ y_C)$ that do not belong to one line. If $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$ are its radius-vectors, the radius-vector $\vec{r}_P$ of any point $P(x,\ y) \in \mathbb{R}^2$ is presented by the unique affine combination

$$\vec{r}_P = \alpha_P \vec{r}_A + \beta_P \vec{r}_B + \gamma_P \vec{r}_C, \tag{3.1}$$

where

$$\alpha_P = \frac{\begin{vmatrix} x & y & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}},\ \beta_P = -\frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}},\ \gamma_P = \frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}. \tag{3.2}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex function, and $z = f^{\text{pla}}_{\{A, B, C\}}(x,\ y)$ be the plane passing through the points $(A,\ f(A))$, $(B,\ f(B))$, and $(C,\ f(C))$ of the graph of f. The plane inequality

$$f(P) \le \alpha_P f(A) + \beta_P f(B) + \gamma_P f(C) = f^{\text{pla}}_{\{A,\ B,\ C\}}(P), \tag{3.3}$$

holds for every point $P \in \operatorname{conv}\{A,\ B,\ C\}$, because for these points the combination in (3.1) is convex. If $P \notin \operatorname{conv}\{A,\ B,\ C\}$, the reverse inequality is not necessarily valid in (3.3).

Lemma 3.1. *Let* $A, B, C \in \mathbb{R}^2$ *be points that do not belong to one line. Let* $D \in \operatorname{conv}\{A, B, C\}$ *be a point. Let* $\alpha, \beta, \gamma \in [0, 1], \delta \in [-1, 1]$ *be coefficients with* $\alpha, \beta, \gamma \geq -\delta$ *of the sum* $\alpha + \beta + \gamma + \delta = 1$.

If

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C + \delta\vec{r}_D,$$

then every convex function $f : \operatorname{conv}\{A, B, C\} \to \mathbb{R}$ *satisfies the inequality*

$$f(P) \leq \alpha f(A) + \beta f(B) + \gamma f(C) + \delta f(D). \tag{3.4}$$

Proof. Let us show that the affine combination $\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C + \delta\vec{r}_D$ is actually the convex combination. Since the point $D \in \operatorname{conv}\{A, B, C\}$, the convex combination $\vec{r}_D = \alpha_D\vec{r}_A + \beta_D\vec{r}_B + \gamma_D\vec{r}_C$ holds with the coefficients $\alpha_D, \beta_D, \gamma_D$ taken from the formulas in (3.2). Therefore, we have

$$\begin{aligned}\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C + \delta\vec{r}_D &= [\alpha(1-\alpha_D) + (1-\beta-\gamma)\alpha_D]\vec{r}_A \\ &\quad + [\beta(1-\beta_D) + (1-\alpha-\gamma)\beta_D]\vec{r}_B \\ &\quad + [\gamma(1-\gamma_D) + (1-\alpha-\beta)\gamma_D]\vec{r}_C. \end{aligned} \tag{3.5}$$

The coefficients in the square brackets are non-negative with the sum equal to 1, so the observed combination is convex, and $P \in \operatorname{conv}\{A, B, C\}$.

If $\delta \geq 0$, the inequality in (3.4) is the Jensen inequality for the four-membered convex combination $\vec{r}_P$.

If $\delta \leq 0$, we use the plane inequality in (3.3) and the affinity of $f_{\Delta}^{\text{pla}} = f_{\{A, B, C\}}^{\text{pla}}$ (Lemma 2.1 is valid for the affine functions of several variables) to obtain

$$f(P) \le f_\Delta^{\text{pla}}(P) = \alpha f_\Delta^{\text{pla}}(A) + \beta f_\Delta^{\text{pla}}(B) + \gamma f_\Delta^{\text{pla}}(C) + \delta f_\Delta^{\text{pla}}(D)$$

$$\le \alpha f(A) + \beta f(B) + \gamma f(C) + \delta f(D),$$

respecting the facts $f_\Delta^{\text{pla}}(A) = f(A)$, $f_\Delta^{\text{pla}}(B) = f(B)$, $f_\Delta^{\text{pla}}(C) = f(C)$ and particularly $\delta f_\Delta^{\text{pla}}(D) \le \delta f(D)$. □

Theorem 3.2. *Let* $A, B, C \in \mathbb{R}^2$ *be points that do not belong to one line. Let* $P_i \in \operatorname{conv}\{A, B, C\}$ *be points. Let* $\alpha, \beta, \gamma, p_i \in [0, 1]$, $\delta \in [-1, 1]$ *be coefficients with* $\alpha, \beta, \gamma \ge -\delta$ *of the sums* $\alpha + \beta + \gamma + \delta = \sum_{i=1}^n p_i = 1$.

If

$$\vec{r}_P = \alpha\vec{r}_A + \beta\vec{r}_B + \gamma\vec{r}_C + \delta\sum_{i=1}^n p_i\vec{r}_{P_i},$$

then every convex function $f : \operatorname{conv}\{A, B, C\} \to \mathbb{R}$ *satisfies the inequality*

$$f(P) \le \alpha f(A) + \beta f(B) + \gamma f(C) + \delta\sum_{i=1}^n p_i f(P_i). \tag{3.6}$$

Proof. The proof coincides with the proof of Theorem 2.4 by applying Lemma 3.1. □

Substituting $\alpha = \beta = \gamma = -\delta = 1/2$ into the formula in (3.6), we get the triangle analogy of the Jensen-Mercer interval inequality in (1.3):

$$f\left(\frac{x_A + x_B + x_C - \sum_{i=1}^n p_i x_i}{2}, \frac{y_A + y_B + y_C - \sum_{i=1}^n p_i y_i}{2}\right)$$

$$\le \frac{f(x_A, y_A) + f(x_B, y_B) + f(x_C, y_C) - \sum_{i=1}^n p_i f(x_i, y_i)}{2}. \tag{3.7}$$

We finish the article by generalizing Theorem 3.2 to simplexes. If $A_1, \ldots, A_{m+1} \in \mathbb{R}^m$ are points such that the vectors $\vec{r}_{A_1} - \vec{r}_{A_{m+1}}, \ldots, \vec{r}_{A_m} - \vec{r}_{A_{m+1}}$ are linearly independent, then the convex hull $\operatorname{conv}\{A_1, \ldots, A_{m+1}\}$ is called the m-simplex with vertices $A_1, \ldots, A_{m+1}$. Geometrically speaking, all the simplex vertices can not belong to the same hyperplane. If $A_j = A_j(x_{j1}, \ldots, x_{jm})$, the radius-vector $\vec{r}_P$ of any point $P(x_1, \ldots, x_m) \in \mathbb{R}^m$ is presented by the unique affine combination

$$\vec{r}_P = \sum_{j=1}^{m+1} \alpha_j \vec{r}_{A_j}, \tag{3.8}$$

where the coefficients α_j can be calculated generalizing determinants of (3.2).

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and $x_{m+1} = f^{\text{hyp}}_{(A_j)}(x_1, \ldots, x_m)$ be the hyperplane (in $\mathbb{R}^{m+1}$) passing through the points $(A_j, f(A_j))$ of the graph of *f*. The hyperplane inequality

$$f(P) \le \sum_{j=1}^{m+1} \alpha_j f(A_j) = f^{\text{hyp}}_{\{A_j\}}(P), \tag{3.9}$$

holds for every point $P \in \operatorname{conv}\{A_j\}$.

Corollary 3.3. *Let $A_1, \ldots, A_{m+1} \in \mathbb{R}^m$ be points that do not belong to one hyperplane. Let $P_i \in \operatorname{conv}\{A_1, \ldots, A_{m+1}\}$ be points. Let α_j, $p_i \in [0, 1]$, $\alpha \in [-1, 1]$ be coefficients with $\alpha_j \ge -\alpha$ of the sums $\sum_{j=1}^{m+1} \alpha_j + \alpha = \sum_{i=1}^{n} p_i = 1$.*

If

$$\vec{r}_P = \sum_{j=1}^{m+1} \alpha_j \vec{r}_{A_j} + \alpha \sum_{i=1}^{n} p_i \vec{r}_{P_i},$$

then every convex function $f : \mathrm{conv}\{A_1, \dots, A_{m+1}\} \to \mathbb{R}$ *satisfies the inequality*

$$f(P) \le \sum_{j=1}^{m+1} \alpha_j f(A_j) + \alpha \sum_{i=1}^{n} p_i f(P_i). \tag{3.10}$$

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